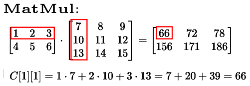
Math

Dot Product: $\mathbf{a}\cdot\mathbf{b} = x_a x_b + y_a y_b + z_a z_b = ||\mathbf{a}||||\mathbf{b}||\cos\phi$

Cross Product: $\mathbf{a}\times\mathbf{b} =$ $(y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b) = ||\mathbf{a}||||\mathbf{b}||\sin(\theta)\mathbf{n}$

MatMul:



For target $c_{i,j}$ iterate over the ith row in the first matrix and the jth column in the second – perform inner product and save.

Right Hand Rule: For axises, x is thumb, y is index, and z is middle finger.

Interpolation: t is how far along the line p_t is from p_0 to p_1 as a percentage between 0 and 1.
$t = (x_t - x_0)/(x_1 - x_0)$ or $t = (y_t - y_0)/(y_1 - y_0)$
$v_t = (1-t)v_0 + tv_1$ **Bi-linear Interpolation** linearly interpolate both sides, then linearly interpolate that.

Cramer's Rule
Given $Ax = b$, where $A \in R^{nxn}$ has a nonzero determinant, then $x_i = \frac{\det(A_i)}{\det(A)}$.. where A_i is A with the ith column replaced with b.

Barycentric Interpolation (Area): From Triangle ABC, point $P(x,y,z)$ can be defined using u, v, w, where $P = uA + vB + wC$. **IMPORTANT:** $u + v + w = 1$.

From World to Bary:

$$P = uA + vB + wC = u\begin{bmatrix}0\\4\\0\end{bmatrix} + v\begin{bmatrix}0\\0\\0\end{bmatrix} + w\begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3w\\4u\\0\end{bmatrix}; \boxed{\begin{bmatrix}u\\v\\w\end{bmatrix} = B\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}y/4\\1-y/4-x/3\\x/3\end{bmatrix}}$$

$\boxed{w = x/3, u = y/4}$
$u + v + w = 1$
$\boxed{v = 1 - u - w = 1 - y/4 - x/3}$

Transformations & Coordinate Systems

Linear Transformation:
$f(\begin{bmatrix}x\\y\end{bmatrix}) = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a_{11}x + a_{12}y\\a_{21}x + a_{22}y\end{bmatrix}$ satisfies:
$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ and $f(c\mathbf{u}) = cf(\mathbf{u})$
In other words, origin is unchanged, straight lines remain straight lines.

Affine Transformation Straight lines remain lines.

2D Rotations
$f(p) = x\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix} + y\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

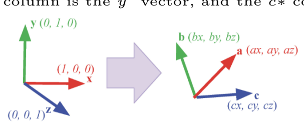
3D Rotations
Note: All orthonormal matrices are rotation matrices.

Rotate around X $f(p) = \begin{bmatrix}1 & 0 & 0\\0 & \cos\theta & -\sin\theta\\0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$

Rotate around Y $f(p) = \begin{bmatrix}\cos\theta & 0 & \sin\theta\\0 & 1 & 0\\-\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$

Rotate around Z $f(p) = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$

For a generic 3D rotation: $R = \begin{bmatrix}ax & bx & cx\\ay & by & cy\\az & bz & cz\end{bmatrix}$ Where the $a*$ column is the coordinates of the object's x' vector, the $b*$ column is the y' vector, and the $c*$ column is the z' vector:

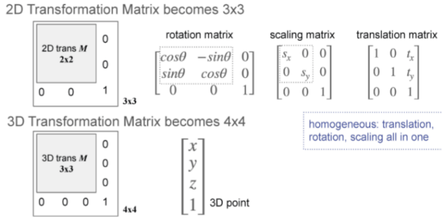


2D Reflection Reflect X: $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ Reflect Y: $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

3D Scaling $\begin{bmatrix}s_x & 0 & 0\\0 & s_y & 0\\0 & 0 & s_z\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}s_x x\\s_y y\\s_z z\end{bmatrix}$

Homogenous Coordinate 2D points represented by (x,y,z), 2D location is $(x/z, y/z) \mid z = 1$ or other value 3D points represented by (x,y,z,w), 3D location is $(x/w, y/w, z/w) \mid w = 1$ or other value Used for perspective projection, z becomes depth value.

Transformation Matrix



Composite Transformation Since scaling, rotation, etc is about origin, operations are not commutive!
Examples To rotate around an object's centre, 1 translate object centre to origin, 2 rotate, 3 translate back.
To scale an object along a non uniform axis, 1 rotate the object to align with a canonical axis, 2 scale object, 3 rotate object back.

Inversions For rotation, scaling, and translating, the inverse of a matrix M^{-1} can be used. For rotation specifically, $M^{-1}_{rotate} = M^{T}_{rotate}$ For scaling, inversion is essentially $1/s$ for your scale factor. For translating, do $-x$.

Coordinate Systems *World/Global Coordinate* Only one, unique. Each model in scene goes through M_{model} to transform from model space to world space.

Camera / Viewing Transformations

Viewing General Steps

1. Model 3D Objects in local space
2. Put 3D object at world coordinates
3. View scene in Camera space
4. Project camera space into cannonical space
5. Transform cannonical space to screen

World Space → Camera Space: M_{cam} Given camera position e, gaze direction g and 'up' direction t, construct basis uwv for camera coordinate system.
$w = -\frac{g}{||g||}, u = \frac{t\times w}{||t\times w||}, v = w \times u$

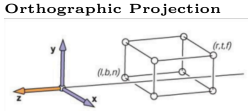
Camera Space → World Space:
$$\begin{bmatrix}1 & 0 & 0 & x_e\\0 & 1 & 0 & y_e\\0 & 0 & 1 & z_e\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_u & x_v & x_w & 0\\y_u & y_v & y_w & 0\\z_u & z_v & z_w & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

World Space → Camera Space: $M_{cam} =$
$$\begin{bmatrix}x_u & y_u & z_u & 0\\x_v & y_v & z_v & 0\\x_w & y_w & z_w & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & -x_e\\0 & 1 & 0 & -y_e\\0 & 0 & 1 & -z_e\\0 & 0 & 0 & 1\end{bmatrix}$$

3×3 Camera Matrix $M_{cam} = [x_{cam} \quad y_{cam} \quad z_{cam}]$
Use column vectors
Step 1 (left matrix): Translates camera position e to origin e. Things originally at e are now at $\vec{0}$.
Step 2 (right matrix): Rotates, maps basis vectors: $u \mapsto (1,0,0), v \mapsto (0,1,0), w \mapsto (0,0,1)$

Cannonical Space
Cannonical space is the space (x,y,z) s.t. $x,y,z \in [-1,1]$.

Orthographic Projection



Given left plane x, l, right plane x, r, top plane y, t, bottom plane y b, near plane n, far plane f. *Recall that f, n are negative z values. See diagram.*
M_{orth} projects the view box defined above onto the canonical space.

$$M_{orth} = \begin{bmatrix}\frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l}\\0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b}\\0 & 0 & \frac{2}{n-f} & -\frac{f+n}{n-f}\\0 & 0 & 0 & 1\end{bmatrix}$$

translates center to origin, scales width of all dimensions to be 2, fitting inside [-1, 1].

Perspective Projection
l, r, b, t now define the near plane XY, with n being the Z. Far plane is only defined by f. **OR** use depth of field: $\theta =$ FOV on y-axis, $ratio = (r-l)/(t-b)$, n is near plane z, f is far plane z.

Frustrum to Box: Recall Homogenous Coordinates. We want frustrum ↦ box s.t. n stays near and f stays far.

$$P = \begin{bmatrix}n & 0 & 0 & 0\\0 & n & 0 & 0\\0 & 0 & n+f & -fn\\0 & 0 & 1 & 0\end{bmatrix}$$

$$M_{per} = M_{orth}P = \begin{bmatrix}\frac{2n}{r-l} & 0 & \frac{l+r}{l-r} & 0\\0 & \frac{2n}{t-b} & \frac{b+t}{b-t} & 0\\0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{n-f}\\0 & 0 & 1 & 0\end{bmatrix}$$

Cannonical Space To View Port (VP)
Diagram of viewport, which has no z depth.
1. Translate x, y center from $(0,0)$ to $((n_x-1)/2, (n_y-1)/2)$. 2. Scale x, y size from $(2,2)$ to (n_x, n_y).

$$M_{vp} = \begin{bmatrix}\frac{n_x}{2} & 0 & 0 & \frac{n_x-1}{2}\\0 & \frac{n_y}{2} & 0 & \frac{n_y-1}{2}\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Rasterization

Definition: finding all pixels on the screen that are occupied by a geometric primitive.

Line Rasterization Given pixels $P_0 = (x_0, y_0), P_1 = (x_1, y_1)$, fill the pixels on the screen between them.
$f(x,y): y = mx + c$
given $m = (y_1 - y_0)/(x_1 - x_0), c = (x_1 y_0 - x_0 y_1)/(x_1 - x_0)$

Implicit line function:: $f(x,y): Ax + By + C = 0$.
Checking sign of $f(x_P, y_P)$ can determine point position relative to line. If $A > 0 \wedge B < 0$, then $f(x_P, y_P) < 0$ means point P is above the line.

Naive Implementation: Use the function $y = mx + c$ and iterate x from x_0 to x_1. No vertical lines!

Non-overlapping Triangles sharing edges: We want to ensure no-double drawing. Assume T_1, T_2 share one edge. Let a be the vertex of T_1 not along this edge. Let b be the coresponding vertex in T_2. Choose offscreen point q. T_1 should be responsible of drawing the edge if q falls on the same side of the edge as a. Likewise for T_2 and b.

Proper Perspective Attribute Attribution: How we set up the above code leads to incorect attribute interpolation when taking perspective into account. This is because we are interpolating on the 2d projection of the triangle and not considering the distace of the 3d space.

Anti-Aliasing
Supersample: Screen Goal is 256x256. We instead render x4, meaning we actually render a 1024x1024 image. Each highresolution pixel is considered a **fragment**. Then, on scale down, we can average the 16 virtual pixels into 1 screen pixel. Can use box filter or a gaussian filter.
Multisample: Screen goal is 256x256. We still rasterize for a higher resolution, but fragment (triangle) colour computation is only calculated once. Then, we sample n times from the pixel area, averaging the samples to get the true pixel colour. **Fragments** here are each dot on a pixel.

Pipeline

Application
Main Program Runs on CPU Defines geometry vertex positions, normals, texture coords, colours, etc. Sets up camera position, orientation, projection volume Sets screen size
Copies data to GPU

Vertex Shader
Per-vertex Computation No transformation: Simply assigns input to output (pass-through shader) Transforming vertex: Apply $M_{proj}M_{cam}M_{model}$ to input Shading: determines vertex color (Gouraud)
GPU performs parallel processing on each vertex

Culling
Backface Culling Removes primitives facing away from camera Look at face normal / right hand rule, face normal points in same direction as face.
View Frustum Culling Removes geometries outside view volume. 6 planes: near, far, left, right, top, bottom. Plane function is $f(p) = n\cdot(p-a) = n\cdot p + n\cdot a = n\cdot p + D = 0$
Test if outside view volume Take bounding box of object, e.g. a sphere with centre c, radius r. Check $f(c)$, c's signed distance to plane, see if it intersects or is within frustrum.
Clipping View volume cuts primitive to avoid drawing out of bounds
Clipping a Line *Plane Function:* $f(p) = n\cdot p + D$ or $f(p) = n\cdot(p-c)$, p is some point, n is the normal, D is a known const, c is a known point on the plane. If $f(p) = 0$, then p is on the plane.
Line Function: $p(t) = \mathbf{a} + t(\mathbf{b}-\mathbf{a})$
Intersection Point Plug p into plane function
$f(p) = n\cdot(\mathbf{a} + t(\mathbf{b}-\mathbf{a})) + D = 0$ Solve for $t = \frac{n\cdot\mathbf{a}+D}{n\cdot(\mathbf{a}-\mathbf{b})}$

Clipping a Triangle
Plane Function: Same as above
Intersection: Assume a, b is on one side, c is on the other side Compute intersection points $\mathbf{A}, \mathbf{B}$ using line clipping method.
Split Triangle $\mathbf{T_1} = \triangle abA, \mathbf{T_2} = \triangle bBA, \mathbf{T_3}$
Throw Away If $f(c) \geq 0$, keep T_3; if $f(c) < 0$, keep $\mathbf{T_1}, \mathbf{T_2}$

Special Case Handle zero-area triangles

Depth Testing

We need to order object rendering so things closer to camera appear 'ontop' of things futher away. Multiple primities can occupy the same fragment.

Painter's Algorithm: Sort primitive by their depths, draw primitives far to near. *Drawbacks:* Sorting is slow, many writes to buffer Occlusion cycle: cases where no correct order appears correct

Color and Z Buffer Two buffers, one for colour and one for depth. Draw primitives as they come in (no sorting), check z buffer (inited with ∞). If the primitive's depth is closer to the camera than what is there (smaller), update the z buffer and override the colour buffer.

Z Fighting: is caused by two primitives sharing the same z value. There are 2^n distinct values that z can be, where n is the number of bits for the depth value. *Precision Formula*:
precision $= (z_{far} - z_{near})/2^b$ We want
precision < max difference between z values *Mitigation Tactices*: Good near far planes, objects not too close together.

Transparency / Alpha We can define a primitive's Transparency as $\alpha \in [0,1]$, color now is RGBA.
src is the colour we want to write, $dest$ is the colour existing in the buffer.
Over Operation: is defined as $\alpha_{src}C_{src} + (1-\alpha_{src})C_{dest}$. This keeps buffer alpha after the operation=1.
Post-multiplication Set dest rgb using the over operation. $C_{src} = (R, G, B)$.
Pre-multiplication Premultiplied alpha has C_{src} already multiplied together with α_{src} when calculating the blending. Thus, we set dest rgb. $C_{src} = (R\alpha_{src}, G\alpha_{src}, B\alpha_{src})$. $C_{dest} = C_{src} + (1-\alpha_{src})C_{dest}$.

Alpha and Depth Test
Zbuffer does not care if the fragment has Transparency – Fragments are not ordered. However- **ORDER matters when dealing with transparency!**. Thus, We draw all opaque objects first using the depth buffer, then use painters algorithm to draw transparent objects.

Mesh

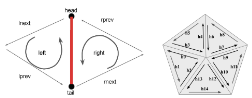
Manifold intuition: Mesh is "watertight", "a small neighborhood around any point could be smoothed out into a bit of flat surface" **Manifold:** Every edge is shared by exactly two triangles. Every vertex has a single, complete loop of triangles around it. **Manifold with Boundary:** Every edge is used by either one or two triangles. Every vertex connects to a single edge-connected set of triangles.
Manifold useful for 2d regular grid, better control of neighboring topology, consistent triangle orientation.

Implicit vs Explicit: Explicit Defines 3D geometry via vertices, edges, and faces. Implicit Defines 3D geometry via a mathematical function
Pros of Explicit Representation: Efficient rendering, direct manipulation, widely supported. **Cons of Explicit Representation:** Memory-heavy, complex storage, explicit connectivity needed. **Pros of Implicit Representation:** Compact for complex shapes, smooth surfaces, ideal for Boolean operations. **Cons of Implicit Representation:** Costly rendering, harder to edit, requires function evaluation.

Euler's Formula: $V - E + F = 2(1-g) \approx 0$ Where F is # of triangles, E is # of edges, V is # of verticies, g is genus (# of holes in surface) Each edge is used 2x, each triangle uses 3 edges, $2E = 3F$. $F = 2V, E = 3V$. V: E: F ≈ 1: 3 :2

Mesh data structures
Separate Triangles (Triangle Soup) It's just an unorganized list of triangles. Storage cost: 72 bytes per vertex (F triangles, 3 vertices, 3 vector components, Euler formula)
Indexed Triangle Mesh Store list of vertices, list of triangle indices separately. Allows for deduplicating vertices, decoupling vertex positions from connectivity. (Blendshapes)
Vertices in Bytes: $V \cdot 3 \cdot 4 = 12V$ Triangles in Bytes: $F \cdot 3 \cdot 4 = 12F \approx 24V$ Total in bytes is 36 bytes / vertex
Triangle Fans List of vertices, list of fan arrays First vertex is center. Each following adjacent pair forms a triangle with center coord.
Triangle Strips Three adjacent vertices form a triangle. Streaming in new vertex. Forget the oldest vertex. Swap order of remaining two vertices for every other triangle (don't swap 0th, swap 1st, etc.)
Storage for fans and strips, vertices is $12V$ (same as ITM), triangle fans / strips array is $(F+2)\cdot 4 = 4F + 8 \approx 8V + 8$. Total is 20 bytes / vertex
Triangle-Neighbor Structure Connectivity info stored in Triangle. 3 vertices in order, 3 neighboring triangles in sync with v. The side from $v[i]$ to $v[x]$ belongs to triangle i. Efficient for adjacency-based operations like smoothing, traversal, and region-growing, and is memory-efficient for static meshes. Lacks granularity at the edge or vertex level, making it unsuitable for operations requiring detailed connectivity or dynamic topology changes. **Traverse**

triangles of vertex v. Pick any triangle t connects to v. Find the neighboring triangle corresponding to v. Set t to be neighboring triangle. Loop until we get back to the start. **Storage:** Triangles: $F \cdot (3+3) \cdot 4 = 24F$ Vertex: $V \cdot (1+3) \cdot 4 = 16V$ Total is $\approx$ 64 bytes / vertex
Winged-Edge Structure Connectivity information stored in edge. 2 vertices of edge, head and tail 2 neighboring triangles, left and right prev and next edges in left triangle, ditto for right triangle. Left tri contains lprev, [tail-head], lnext, . Right tri contains rprev, [head-tail], rnext. **Traversal given vertex:** Follows prev (aka always pick the edge to the left), edges around v visited in ccw order. **Traversal given face:** Stays on the side where face is, visit edges of face in ccw order. **Storage** Edge in bytes: $E \cdot 8 \cdot 4 = 32E$ (8 references, we can remove prevs to save space and just do 2 nexts instead) Triangle in bytes: $F \cot 1 \cdot 4 = 4F$ Vertex in bytes: $V \cdot (1+3) \cdot 4 = 16V$ (1 edge ref, xyz) Total: $\approx 120V$.

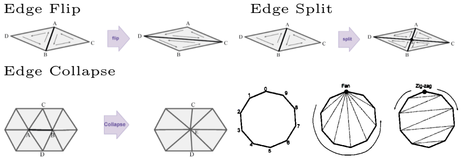


Half-Edge Structure Halfedge: VertexRef v (vertex half-edge points to) TriangleRef t (triangle that the halfedge is on) HalfEdgeRef prev, next (HalfEdgeRef pair
Storage Half-edge in bytes: $(E \cdot 2) \cdot 5 \cdot 4 = 40E$ Triangle in bytes: $F \cdot 1 \cdot 4 = 4F$ Vertex in bytes: $V \cdot (1+3) \cdot 4 = 16V$ Total: $\approx 144V$

Mesh Editing

Edge Flip Edge Split

Edge Collapse



Subdivision

Split: insert new vertices **Interpolate:** recompute positions for original and new vertices

Loop Subdivion

New vertices are added at positions of weighted sum of original vertices on neighboring faces. For interior, $v = 3 \cdot (A+B)/8 + (C+D)/8$ where v lies on the edge between A and B. For boundary, $v = (A+B)/2$. Original vertices are reweighted based on original neighboring vertices. $v_{new} = (1-n\beta)v_{original} + \beta \sum v_{neighbor}$. Where n is v's degree. Implementation: split each triangle edge in any order, flip any edge connecting a new and an original vertex.

Linear Subdivision

Input: m-gon. Output: m quads per m-gon. Face vertex: average of face corner vertices Edge vertices: average of two vertices at ends of edge Vertex from original mesh: keep original.

Catmull-Clark Subdivision

Face vertex position is average of all original face corner vertex positions Edge vertex position is average of two end vertices and two face vertices Original vertex position is change according to $\frac{Q+2R+(n-3)S}{n}$ Where Q is average position of face vertices neighboring v, R is average position of all edge vertices connected to v, S is original vertex position of v, n is v's degree.

Quadric Error Simplification

Given a plane $ax + by +_c z + d = 0$, let

$$u = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ then, } K_{plane} = \begin{bmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix}$$

$K_{vert} = \sum K_p \forall$adj. planes and $K_{edge-ij} = K_i + K_j$. The metric to minimize is $o^T k_{ij} o$, the distance from $o(x, y, z)$ to edge ij. To solve, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}^{-1} \begin{bmatrix} -k_{14} \\ -k_{24} \\ -k_{34} \end{bmatrix}$

Isotropic Remeshing

Let L be mean edge length of input mesh If an edge is too long ($> 4/3L$), split it If an edge is too short ($< 4/5L$), collapse it If flipping an edge improves degree of neighboring vertices, flip it: Deviation is $\sum_A^D |deg(X) - 6|$, if deviation drops after a flip then flip the edge. Move vertex positions towards average of their neighbors: For vertex P, computer centroid C of all neighbours. Movement vector is $v = C - P$, move by $1/8v$. Ignore movement in normal direction N by using $v - N(N \cdot v)$ Vertex normal is weighted sum of face normals $N = \sum_{neighbortrianglet} t.area * t.normal = \sum cross(t.edge1, t.edge2)$ Then normalize

Blendshapes

This is trivial, just calculate offsets with $o_i = (bs_i - base)$, and build final blendshape with $shape = base + \sum w_i \cdot o_i$.

Texture

Projections

Planar Projection $\begin{bmatrix} u \\ v \\ * \\ 1 \end{bmatrix} = M_t \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ Where M_t is affine transformation matrix.

Spherical Coordinates

$u = \frac{[\pi + atan2(y,x)]}{2\pi} \quad (atan2(\ldots) = \phi)$

$v = \frac{[\pi - acos(\frac{z}{\sqrt{x^2+y^2+z^2}})]}{\pi} \quad (acos(\ldots) = \theta)$

$atan2 \in [-\pi, \pi]$, $acos \in [0, \pi]$ Has issues for oddly shaped objects (eg cylinders), areas with large r are sparse vs areas with small r.

Cylindrical Coordinates

$u = \frac{[\pi + atan2(y,x)]}{2\pi} \quad atan2(\ldots) = \phi$

$v = \frac{1+z}{2}$, $z \in [-1, 1]$

Cubemap (Skybox)

$u = \frac{1+y/|x|}{2}$, $v = \frac{1+z/|x|}{2}$ where $|x| > |y|, |x| > |z|, x > 0$

Seams If we don't handle seams, wrapped texture will have triangles "wrap" across the whole texture, making a bad seam where the texture should repeat. Instead, we duplicate the vertices on the left and right edges, such that the left seam's edges only apply to the triangles on it's right, and the right seam's edges only apply to triangles on it's left.

Barycentric Interpolation

Get barycentric of p inside triangle $b1, b2, b3$. Interpolate uv for p: $u_p = b1 \cdot u_a + b_2 \cdot u_b + b_3 \cdot u_c$

Sampling

Nearest: Self-explanatory **Bilinear:** Also kinda self-explanatory, see interpolation on page 1.

Wrapping

UV is in range [0,1], deals with how to handle stuff outside of range. Clamp: use edge colour for stuff outside range Tile: repeatedly sample the texture (loop it)

Mipmap

Take the longest length l, the mipmap level to take is $\log_2 l$. Level 0 is base image, each texel in level k image is avg of $2^k \times 2^k$ texels in original texture.

Maps

Diffuse Map: Stores colours, RGB **Normal Map:** Stores normal vectors to determine bounce directions, 3 channels **Displacement Map:** Changes vertex positions by x amount in the normal direction, 1 channel, slow. The shadow will also have bumps **Bump Map:** Scales normals by x amount in the normal direction, does not change vertex positions, 1 channel, fast. The shadow won't have bumps. **Shadow Map:** Stores distance to first point hit by each light in a preprocessing step (z buffer). During rendering, calculate each fragment's distance from light and illuminate it if it's less than or equal to the stored value. Otherwise, it is in shadow.

Light

2D Angle: $l/r(radians)$ 3D Angle: $A/r^2(steradians)$ **Radiant Flux:** Power (W) **Irradiance:** Power per square unit of area ($\mathrm{W\,m^{-2}}$) **Radiance:** Power per square unit of area per direction ($\mathrm{W\,m^{-2}\,sr^{-1}}$) **Radiant intensity:** Power per direction ($\mathrm{W\,sr^{-1}}$)

Point light irradiance: $E = \frac{\Phi}{4\pi r^2}$ Fractional area on sphere $\frac{2\pi(1-cos\varphi)}{4\pi}$ where φ is cone half-angle

$F_{att} = \frac{1.0}{K_c + K_l \cdot d + K_1 \cdot d^2}$ Spotlight soft edges, inner cutoff $\varnothing$, outer cutoff γ

Material

BRDF (Bidirectional Reflectance Distr. Func.)

$L(P, \omega_0) = L_e(P, \omega_0) + \int_\Omega f_r(P, \omega_i, \omega_0) L_i(P, \omega_i) \cos\theta_i \, d\omega_i$
components $L(P, \omega_0)$: Outgoing radiance @ P in ω_0 direction $L_e(P, \omega_0)$: Emitting light radiance @ P in ω_0 direction if light source, else 0 $\int_\Omega$: Sum over all incoming direction $f_r(P, \omega_i, \omega_0)$: How incoming light at point P in ω_i direction is reflected in ω_0 direction $L_i(P, \omega_i) \cos\theta_i$: Incoming light radiance at point P in ω_i direction, weighted by $\cos\theta_i$.

Diffuse

light direction $l = light_{pos} - vertex_{pos}$
`diffuse_light_clr * diffuse_mat_clr * max(0, dot(n, l)) / r^2`
$(n \cdot l = \cos\theta)$

Specular

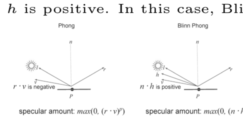
Exponent: lower p is more diffused, higher p is more reflective
Specular OpenGL: Find reflected light direction $r = -l + 2(l \cdot n)n$. Vertex direction P_p, camera position P_c, find view direction $v = norm(P_c - P_p)$. Color on object =
`specular_light_clr * specular_material_clr * max(0, dot(v, r)^p)`

Phong

Phong shading is the sum of ambient, diffuse, and specular lighting. (Multiplied by I/r^2 to simulate light falloff)
$L(P, v) = [k_a + k_d \cos\theta + k_s(\cos\rho)^p] \cdot \frac{I}{r^2} =$
$[k_a + k_d(n \cdot l) + k_s(r \cdot v)^p] \cdot \frac{I}{r^2}$ In OpenGL:
ambient_light_clr * ambient_material_clr +
diffuse_light_clr * diffuse_material_clr * max(0, n·l) +
specular_light_clr * specular_material_clr * max(0, $v \cdot r)^p$

Blinn-Phong

Instead of using the viewing angle for specular, we use the half-vector h, which is halfway between the view and light directions. ($\frac{l+v}{||l+v||}$). When adding specular it is now specular_light_clr + specular_material_clr * max(0, $n \cdot h)^p$. Blinn-Phong can produce bad results when $r \cdot v$ is negative, but $n \cdot h$ is positive. In this case, Blinn-Phong will be too bright.



Gouraud

Instead of computing color based on lighting in the *fragment shader step*, Gouraud does it using *vertex positions* using vertex colors during the vertex shader.

Ray Tracing

Ray representation: $r = o + td$
Ray Gen left, right, top, bottom are coords defining bounds of view volume. view volume is n_x px wide, n_y px high.
pixel(i, j) → (u, v, w) coord
$u = left + (i + 0.5) * (right - left)/n_x$;
$v = bottom + (j + 0.5) * (top - bottom)/n_y$; $w = -fl$
Perspective $o = e$ (e is camera coords); $d = u\vec{u} = v\vec{v} - fl\vec{w}$ (uvw is camera view coords, w is away from looking location)
Orthographic $o = e + u\vec{u} + v\vec{v}$ $d = -\vec{w}$
Ray-Object Intersection Sphere Point $\mathbf{p}$ on ray: $\mathbf{p} = \mathbf{o} + t\mathbf{d}$ Point $\mathbf{p}$ on sphere centred at $\mathbf{c}$ with radius R:
$(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) - R^2 = 0$ Intersection points:
$(\mathbf{o} + t\mathbf{d} - \mathbf{c}) \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{c}) - R^2 = 0$ Solve for t:

$$t = \frac{-\mathbf{d} \cdot (\mathbf{o} - \mathbf{c}) \pm \sqrt{(\mathbf{d} \cdot (\mathbf{o} - \mathbf{c}))^2 - (\mathbf{d} \cdot \mathbf{d}) \cdot ((\mathbf{o} - \mathbf{c}) \cdot (\mathbf{o} - \mathbf{c}) - R^2)}}{\mathbf{d} \cdot \mathbf{d}}$$

Normal $\mathbf{n}$ at hit point $\mathbf{p}$: $(\mathbf{p} - \mathbf{c})/R$
Triangle Representation of $\mathbf{p}$:
$\mathbf{p} = \alpha\mathbf{A} - \beta\mathbf{B} + \gamma\mathbf{C} = (1 - \beta - \gamma)\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$ Point $\mathbf{p}$ on triangle **ABC**: $\mathbf{p} = \mathbf{A} + \beta(\mathbf{B} - \mathbf{A}) + \gamma(\mathbf{C} - \mathbf{A})$ Intersection point: $\mathbf{o} + t\mathbf{d} = \mathbf{A} + \beta(\mathbf{B} - \mathbf{A}) + \gamma(\mathbf{C} - \mathbf{A})$

$$\begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} x_A - x_B & x_A - x_C & x_d \\ y_A - y_B & y_A - y_C & y_d \\ z_A - z_B & z_A - z_C & z_d \end{bmatrix}^{-1} \begin{bmatrix} x_A - x_o \\ y_A - y_o \\ z_A - z_o \end{bmatrix}$$

Rectangle (this is for xy plane) Point p on axis aligned rect: $z_p = k$ Intersection: $z_p = z_o + tz_d = k$ Solve for t: $t = (k - z_o)/z_d$ Plug back, check $x_p \in [x_0, x_1], y_p \in [y_0, y_1]$ so it's in rect.
Shading Lambertian:
$res = diffuse \cdot I \cdot max(0, n \cdot l)/(dist^2)$ diffuse is obj colour, I is light colour, dist is abs(lpos - p), l is (lpos - p) / dist.
Specular: $l = (lpos - p)/|lpos - p|$;
$h = (l + (-d))/|l + (-d)|$;
$res = specular \cdot I \cdot max(0, pow(h\dot{n}, 5))$
Shadows shadow_ray: $p + tl$ dist: $|lpos - p$ $t \in [\epsilon, dist]$ if $t < dist$ then it hit an object before the light and thus is in shadow.

Extra

Spatial Data Structure

Used to quickly locate objects in space **Binary space partition (BSP)** While every object is not in its own region, split the space in half Hyperplanes; for higher dimensions (3D), splits space similarly to Binary space partitioning. Splitting hyperplanes can be completely arbitrary
Autopartition Candidates for splitting are based on input primitives; when we have inputs, we just extend the planes further to split space. **K-D Tree** K means number of dimensions. 2D example: we split along the median of the X values, then we split along the Y medians of the 2 resulting partitions, then X, etc, etc. until every point is a median.
Octree / Quadtree Octree: Divide 3d space into 8 quadrants, keep dividing the resulting space until each has <=1 point in each cell **Quadtree**: Divide 2d space into 4 quadrants, keep dividing until each space has <=1 point in each cell

Curves

Desirable Properties Continuity: n^{th} derivative remains the same. C^0: lines connect, C^1: velocity remains the same, C^2: acceleration remains the same. **Locality:** Local - one control point only impacts a small piece of the curve. Global - one control point impacts the shape of the whole curve
Interpolation: Interpolation - Curve passes through all control points. Approximation - Curve does not pass through all control points, but control points determine curve. Ideally we want at least C^2 continuity, locality, and interpolation.
Disadvantages of Single Piece Polynomial Curves

- Lack of locality. Changes to points affect the whole curve.
- Runge's Phenomenon: Oscillation with polynomial interpolation causes increasing wiggles and overshoots near the outer control points.

Cubic Spline

- Defined as $a_0 + a_1 t + a_2 t^2 + a_3 t^3$.
- Properties: Locality, C^2 continuity, Interpolates the control points.

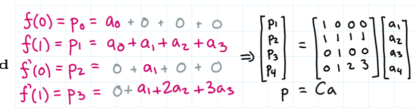
Cardinal Spline

- Defined by $n + 1$ control points.
- Uses positional and derivative constraints.
- Derivatives calculated using a tension constant c and the positions of neighboring points: $F(0) = p_1$ $F'(0) = (1 - c)(p_2 - p_0)/2$
- Catmull-Rom is a special case of Cardinal Spline where $c = 0$, implying $F'(0) = p_2 - p_0)/2$

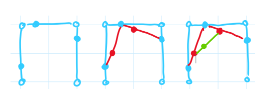
Properties of Different Curve Types TC = Total Constraints

Curve Type	Continuity	Locality	TC
Single Polynomial	C^n	Global	–
Hermite Cubic	C^1	Local	$4n$
Cardinals	C^2	Local	$4n$
Catmull-Rom	C^2	Local	$4n$

Deriving Spline Basis for Cubic Curves
Recall the cubic functions: $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$.
$f'(t) = a_1 t + a_2 t + 3a_3 t^2$.
We also have the Hermite equations for a segment:



Take $p = Ca$, rearrange $C^{-1}p = a$.
Now we have expressions for a_i in terms of p. We can replace all $a's$ in the original $f(t)$ to be in terms of p. Rearrange that to collect like terms of p, we get a basis.
Hermite Basis: $f(t) =$
$p_1(2t^3 - 3t + 1) + p_1(-2t^3 + 3t^2) + p_2(t^3 - 2^2 + t) + p_3(t^3 - t^2)$
Where $p_1 = p_0, p_2 = p_1, p_3 = p_0', p_4 = p_1'$
Deriving Bezier Basis
Depending on the bezier 'degree' n, we can calculate it via $b_{i,n} = \frac{n!}{i!(n-i)!}(t^i)(1-t)^{n} - i$.
$b_{0,3} = (1-t)^3$ $b_{1,3} = t(1-t)^2$ $b_{2,3} = t^2(1-t)^1$ $b_{3,3} = t^3$
Computing Positions on the Curve Using Spline Basis
Given user constraints, compute positions using the basis functions b and constraints p:
$F(t) = p_0 b_0(t) + p_1 b_1(t) + p_2 b_2(t) + p_3 b_3(t)$
Computing Positions on Bezier Curves Using the de Casteljau Algorithm Define the 'control polynomial', then connect the intersections between the points.



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1