Trigonometry

$$\csc\theta = 1/\sin\theta$$
$$\sec\theta = 1/\cos\theta$$
$$\sin(\pi/2-\theta) = \cos\theta$$
$$\cos(\pi/2-\theta) = \sin\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$\cos(2\theta) = 1 - 2\sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$
$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$
$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$
$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$
$$\sin\theta\sin\phi = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$$
$$\cos\theta\cos\phi = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}$$
$$\sin\theta\cos\phi = \frac{\sin(\theta+\phi)+\sin(\theta-\phi)}{2}$$
$$\cos\theta\sin\phi = \frac{\sin(\theta+\phi)-\sin(\theta-\phi)}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Models

Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_m),\quad T(t_0) = T_0$$

where $T(t)$ is temperature of body at time t, $k > 0$ is coefficient of heat transfer (constant) and T_m is temperature of surroundings (assumed to be an infinite reservoir that can absorb any amount of heat without changing its temperature).

$$T(t) = T_m + (T_0 - T_m)e^{-k(t-t_0)}$$

Draining a Tank

Rate of volume of liquid leaving the tank is proportional to, $A_h v$, where A_h is area of hole,

$$\frac{dV}{dt} = -A_h\sqrt{2gh}$$

Finally, volume of liquid in tank is $V(t) = A_w h$, where A_w is (constant) area of tank

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh},\quad h(t_0) = h_0$$

$$h(t) = \left(h_0^{1/2} - \frac{A_h\sqrt{2g}}{2A_w}(t - t_0)\right)^2$$

Population Growth

$$\frac{dP}{dt} = rP,\quad P(t_0) = P_0 \quad \text{(one parameter model: } r\text{)}$$

where $P(t)$ is the population at time t and r is the rate of population growth.
Linear equation can be solved easily by integration. Gives (disastrous!) exponential population growth:

$$\int_{P_0}^{P}\frac{dP}{P} = \int_{t_0}^{t} r\,dt \implies \log\left(\frac{P(t)}{P_0}\right) = r(t - t_0)$$

$$P(t) = P_0 e^{r(t-t_0)}$$

Realistic Population Growth

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \quad \text{(two parameter model: } r, K\text{)}$$

where r is rate of population growth and K is the carrying capacity (the long time stable equilibrium population). This is also an autonomous ODE.

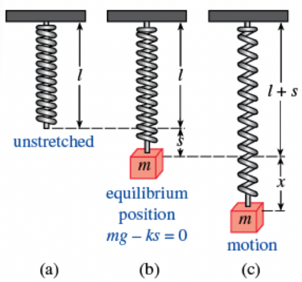
$$P(t) = P_0\frac{K}{P_0 + (K - P_0)e^{-rt}}$$

Beem deflection

TABLE 3.9.1.

Ends of the Beam	Boundary Conditions
Embedded	$y = 0, y' = 0$
Free	$y'' = 0, y''' = 0$
Simply supported or hinged	$y = 0, y'' = 0$

Hook's Law of Springs



Spring motion when deformed by gravity $F_g = -mg$, where m is mass of suspended weight. $s > 0$ is equilibrium displacement under gravity (i.e. gravitation force balances restoring spring force). x is additional displacement from equilibrium position (spring is stretched or compressed).

- **Hooke's law**: $F_s = -k(s + x)$.
  (additional elongation x compared to equilibrium elongation s under gravity)
  Note: $x > 0$ indicates *downward* displacement ("Reversed coordinates"!)
- Force exerted on mass m by gravity: $F_g = mg$. (Positive sign due to reversed coordinates!)
- **Newton's second law**: $F = ma = m\ddot{x}$.

Force balance:
$$m\ddot{x} = F_s + F_g \implies m\ddot{x} = -k(s+x) + mg \implies -kx + \underbrace{(mg - ks)}_{=0}$$

$$\implies \ddot{x} + \frac{k}{m}x = 0$$

$$\implies \ddot{x} + \omega^2 x = 0$$

where $\omega = \sqrt{k/m}$ is **natural frequency** of oscillations.

$$x(t) = c_1\cos\omega t + c_2\sin\omega t$$

Simple harmonic motion with ***angular frequency*** ω, ***period*** $T = 2\pi/\omega$ and ***frequency*** $1/T = \omega/(2\pi)$.

Simple harmonic motion is also known as ***free vibration*** because dynamical system is homogeneous (unforced, apart from gravity).

Equation of motion for **simple harmonic motion**

$$x(t) = c_1\cos\omega t + c_2\sin\omega t$$

Is often written in an equivalent form involving two different constants: the ***amplitude*** A and ***phase*** ϕ

$$x(t) = A\cos(\omega t - \phi)$$

where $A = \sqrt{c_1^2 + c_2^2}$ and $\phi = \arctan(c_1/c_2)$.

Dampened spring system

$$mx'' = -kx - \beta x'$$

where $\beta > 0$ is damping constant.

$$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$$

where $2\lambda = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$.

Auxiliary equation is: $r^2 + 2\lambda r + \omega^2 = 0$ with solutions $r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$.

3 Cases:

1. **Overdamped:** $\lambda^2 - \omega^2 > 0$ (distinct real roots: exponential decay of motion as $t \to \infty$).
2. **Critically damped:** $\lambda^2 - \omega^2 = 0$ (repeated real root: any slight decrease in damping → oscillations).
3. **Underdamped:** $\lambda^2 - \omega^2 < 0$ (complex conjugate roots: oscillations with decreasing amplitude).

Can also apply external forcing:

Suppose we apply ***external force*** $f(t)$ to suspended mass (e.g. push or pull it).

Dynamical system is now nonhomogeneous:

$$m\ddot{x} + \beta\dot{x} + kx = f(t)$$

or, dividing by m,

$$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = F(t)$$

where $F(t) = f(t)/m$.

$$L[y] = y'' + P(x)y' + Q(x)y = f(x)$$

Thus, given complementary function $y_c = c_1y_1 + c_2y_2$, particular solution is

$$y_p = u_1y_1 + u_2y_2$$

where

$$u_1(x) = \int\frac{W_1(x)}{W(x)}dx = -\int\frac{y_2(x)f(x)}{W(x)}dx,\quad u_2 = \int\frac{W_2(x)}{W(x)}dx = \int\frac{y_1(x)f(x)}{W(x)}dx$$

Cauchy Euler Equations

General soloution is $y = x^m$:

$$L[y] = ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = g(x)$$

Auxiliary equation: $am^2 + (b-a)m + c = 0$

Case 1: distinct real roots $m_1, m_2 \in \mathbb{R}$

$$y_c(x) = c_1x^{m_1} + c_2x^{m_2}$$

Case 2: repeated root $m \in \mathbb{R}$

$$y_c(x) = c_1x^m + c_2x^m\log x$$

Case 3: complex conjugate roots $\alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$

$$y_c(x) = x^\alpha[c_1\cos(\beta\log x) + c_2\sin(\beta\log x)]$$

Partial differential equations

variations

Consider PDE with constant coefficients A, B, C, D, E, F:

$$A\frac{\partial^2u}{\partial x^2} + B\frac{\partial^2u}{\partial x\partial y} + C\frac{\partial^2u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x,y)$$

We classify PDE as one of three types according to value of $B^2 - 4AC$

$$B^2 - 4AC \begin{cases} > 0 & \text{hyperbolic (e.g. wave equation),} \\ = 0 & \text{parabolic (e.g. heat equation),} \\ < 0 & \text{elliptic (e.g. Poisson equation).} \end{cases}$$

For example, if there is heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m, then the heat equation (13) is

$$k\frac{\partial^2u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t},$$

where h is a constant. In (14) the function F could represent the various forces acting on the string. For example, when external, damping, and elastic restoring forces are taken into account, (14) assumes the form

$$a^2\frac{\partial^2u}{\partial x^2} + \underbrace{\overset{\text{external force}}{f(x,t)} - \overset{\text{damping}}{c\frac{\partial u}{\partial t}} - \overset{\text{restoring force}}{ku}}_{F(x,t,u,u_t)} = \frac{\partial^2u}{\partial t^2}. \qquad \textbf{(15)}$$

heat equation

$$k\frac{\partial^2u}{\partial x^2} = \frac{\partial u}{\partial t}$$

wave equation

$$a^2\frac{\partial^2u}{\partial x^2} = \frac{\partial^2u}{\partial t^2}$$

laplace equation

$$\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = \rho(x, y)$$

We found general solution

$$Y(y) = c_3\cosh\frac{n\pi y}{a} + c_4\sinh\frac{n\pi y}{a}$$

Imposing BC $Y(0) = 0 \implies c_3 = 0 \implies$ eigenfunction solutions compatible with BC are

$$Y_0(y) = y,\ n = 0,\quad Y_n(y) = \sinh\frac{n\pi y}{a},\ n = 1, 2, 3, \ldots$$

Finally, using $u_n(x,y) = X_n(x)\,Y_n(y)$ we find the complete set of eigenfunctions

$$u_0(x,y) = y,\quad n = 0 \ \text{and}\ u_n(x,y) = A_n\cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a},\quad n = 1, 2, 3, \ldots$$

Linear combinations of eigenfunctions give general solution

$$u(x,y) = A_0y + \sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

How do we impose BC $u(x,b) = f(x)$?

By computing coefficients A_n such that

$$u(x,b) = f(x) = A_0b + \sum_{n=1}^{\infty}\left(A_n\sinh\frac{n\pi b}{a}\right)\cos\frac{n\pi x}{a}$$

Need to compute coefficients A_n from Fourier series for $f(x)$ by setting $y = b$:

$$u(x,b) = f(x) = A_0b + \sum_{n=1}^{\infty}\left(A_n\sinh\frac{n\pi b}{a}\right)\cos\frac{n\pi x}{a}$$

$$A_0b = \frac{1}{2}\frac{2}{a}\int_0^a f(x)\,dx \implies A_0 = \frac{1}{ab}\int_0^a f(x)\,dx \quad \text{Note extra factor } \tfrac{1}{2} \text{ for } A_0 \text{ cosine series coefficient!}$$

$$A_n\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx \implies A_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f(x)\cos\frac{n\pi x}{a}dx,\quad n = 1, 2, 3, \ldots$$

$$f(x) = A_0 b + \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi b}{a} \right) \cos \frac{n\pi x}{a},$$ where coefficients A_n are given by

$$A_0 = \frac{1}{ab} \int_0^a f(x)\, dx, \qquad A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \cos \frac{n\pi x}{a}\, dx, \quad n = 1, 2, 3, \ldots$$